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Topological Field Theories

- What does it mean to be a topological theory? (Schwarz + Witten types)
- A mathematical description of a topological field theory
- A physical example of a topological field theory

Disclaimer: We are going to have to use some very sophisticated ideas that we will not have time to develop in detail (maybe at a later time).

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What does it mean to be a topological theory?

When we specify a physical theory we must include its d.o.f. and their interactions. Part of specifying d.o.f. is providing the space(time) upon which they do their thing.

In specifying a space(time) we have 2 almost independent features:

I can mess w/ things far away and preserve geometry in a neighborhood

Geometry - scale, distance, local, usually described by a metric (w/ coordinates)

stretching changes it

gives us clocks and rulers to define distance

cares about everywhere all at once

Topology - doesn't care about scale, global, usually described by characteristic classes

stretching (but not tearing) does not change it

sets of topologically equ. geometries sharing common features

Most physical theories care about both simultaneously. However it is possible to have situations which only care about the latter. There are 2 ways to do this:

1. If the action S defining a classical (through EL) or quantum (through PI) theory does not contain reference to a metric, then it clearly does not change with changing metrics, so the theory is a Schwarz-type topological (field) theory.

2. On the other hand you may start w/ a theory defined using a metric $g_{\mu\nu}$ and then if it posesses a nilpotent symmetry, i.e. $\delta S = 0$ w/ $\delta^2 = 0$ then (assuming a few other conditions) one can demonstrate that a class of its correlation functions are metric ind. These are called Witten-type TFTs.

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So what does a topological field theory look like?

Consider electromagnetism in $\mathbb{M}^4$: $S_{EM} = -\frac{1}{4}\int F_{\mu\nu}F^{\mu\nu}\, d^4x$ w/ $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$A_\mu = \begin{pmatrix} V \\ \vec{A} \end{pmatrix}$ ← scalar pot. ← "vector" pot.

$F_{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & B^3 & -B^2 \\ E^2 & -B^3 & 0 & B^1 \\ E^3 & B^2 & -B^1 & 0 \end{pmatrix}$

At first glance this does not seem to incorporate a metric, e.g. $\eta_{\mu\nu}$ (or $\eta^{\mu\nu}$).
But recall that the basic ingredient is A_μ and $\frac{\partial}{\partial x^\mu} = \partial_\mu$, so what is $F^{\mu\nu}$? $F^{\mu\nu} = \underbrace{\eta^{\mu\lambda}\eta^{\nu\delta}}_{\text{not one but two (inverse) metric factors}} F_{\lambda\delta}$

But wait, it gets gooder because the measure of integration $\int d^4x$ comes secretly armed w/ a factor $\sqrt{-\det\eta_{\mu\nu}} = +1$ which is important to remember if we a) go to curved space b) use curvilinear coordinates on $\mathbb{M}^4$

⇓

$\sqrt{-g} \equiv \sqrt{-\det g_{\mu\nu}} \neq +1$ in general

Why the $\sqrt{-g}$? Recall that under a coordinate transformation $\underbrace{d^4x}_{x^\mu} = \underbrace{d^4x'}_{x^{\mu'}} \underbrace{J}_{\det\left(\frac{\partial x^\mu}{\partial x^{\mu'}}\right)}$ or $d^4x' = \frac{1}{J}\, d^4x$

But this means that something as simple as a volume $\int d^4x$ is <u>not</u> coord. invariant.
To fix it we note that the metric itself transforms as $g_{\mu\nu} \to g_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu}$

$\Rightarrow \det g \to \det g' = J^{-2}\det g$

or $g \to g' = J^{-2} g$

So finally: $\int d^4x\sqrt{-g} \to \int d^4x'\sqrt{-g'} = \int \frac{1}{J}\, d^4x\, \sqrt{-J^{-2}g} = \underbrace{\int d^4x\sqrt{-g}}_{\text{so is coord. inv.}}$

In the end good old $S_{EM} = -\frac{1}{4}\int d^4x\, F_{\mu\nu}F^{\mu\nu}$ depends on $\eta_{\mu\nu}$ (or $g_{\mu\nu}$ in general).

But...

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From our discussion of differential forms we might recall that $\int_{\Sigma_p} A^{(p)}$ where $A^{(p)}$ is a p-degree form and Σ_p is a p-dimensional manifold $\underline{\underline{\text{is}}}$ coordinate invariant.

Recall this arises because the anti-commuting basis of forms in $A^{(p)} = \frac{1}{p!} \overbrace{A_{\mu_1 \cdots \mu_p}}^{\text{components}} \underbrace{dx^{\mu_1} dx^{\mu_2} \cdots dx^{\mu_p}}_{\text{basis forms}}$ forms a well defined integration measure over an oriented p-dim. volume.

So if we take $p=4$ and let Σ_p be our spacetime $\mathbb{M}^4$, then we can write a new term for S:

$$S = S_{EM} + \underbrace{\int A^{(4)}}_{\text{what might this look like?}}$$

Well we would like to build it in terms of A_μ and ∂_μ.

The first thing to note is that A_μ are the components of a 1-form $A^{(1)} = A_\mu dx^\mu$, while $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ are the components of a 2-form $F^{(2)} = \frac{1}{2} F_{\mu\nu} dx^\mu dx^\nu$ (recall form components $\underline{\underline{\text{must}}}$ be anti-symmetric under index exchange, i.e. $F_{\mu\nu} = -F_{\nu\mu}$).

The basis $dx^\mu dx^\nu$ should also anti-commute so one way to remember this is w/ $\epsilon^{\mu\nu} d^2x$.

So we have a 1-form and a 2-form at our disposal, but recall that the wedge product $A^{(p)} \wedge B^{(q)} = C^{(p+q)}$ where in terms of components $C_{\mu_1 \cdots \mu_{p+q}} = (A \wedge B)_{\mu_1 \cdots \mu_{p+q}} = \frac{(p+q)!}{p!\, q!} A_{[\mu_1 \cdots \mu_p} B_{\mu_{p+1} \cdots \mu_{p+q}]}$

One option for $A^{(4)}$ is

$$F^{(2)} \wedge F^{(2)} = \underbrace{\frac{(2+2)!}{2!\,2!}}_{6} \underbrace{F_{[\lambda\rho} F_{\mu\nu]}}_{\text{antisymmetrize over 4 indices}} \epsilon^{\lambda\rho} d^2x\, \epsilon^{\mu\nu} d^2x$$

$$= 6 \frac{1}{24} \left(F_{\lambda\rho} F_{\mu\nu} + 23 \text{ permutations} \right) \epsilon^{\lambda\rho} d^2x\, \epsilon^{\mu\nu} d^2x$$

$$= F_{\lambda\rho} F_{\mu\nu} \epsilon^{\lambda\rho\mu\nu} d^4x$$

Notice $\epsilon^{\lambda\rho\mu\nu} \neq \epsilon^{\lambda\rho} \epsilon^{\mu\nu}$

So we can add:

$$\frac{1}{4} \int d^4x\, \epsilon^{\lambda\rho\mu\nu} F_{\lambda\rho} F_{\mu\nu} = \frac{1}{4} \int d^4x\, \epsilon^{\lambda\rho\mu\nu} (\partial_\lambda A_\rho - \partial_\rho A_\lambda)(\partial_\mu A_\nu - \partial_\nu A_\mu)$$

$$= \frac{1}{4} \int d^4x\, \epsilon^{\lambda\rho\mu\nu} (\partial_\lambda A_\rho \partial_\mu A_\nu - \partial_\lambda A_\rho \partial_\nu A_\mu - \partial_\rho A_\lambda \partial_\mu A_\nu + \partial_\rho A_\lambda \partial_\nu A_\mu)$$

$$= \int d^4x\, \epsilon^{\lambda\rho\mu\nu} (\partial_\lambda A_\rho)(\partial_\mu A_\nu)$$

This guy does $\underline{\underline{\text{not}}}$ depend on the metric! Essentially, we replaced index contraction w/ $g^{\lambda\mu} g^{\rho\nu}$ by contraction w/ $\epsilon^{\lambda\rho\mu\nu}$, and the $\sqrt{-g}$ term is not needed since forms already have the correct transformation built in.

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relative weighting of terms

In the end we can consider: $S = -\frac{1}{4}\int d^4x \sqrt{-g}\, g^{\lambda\mu} g^{\rho\nu} F_{\lambda\rho} F_{\mu\nu} + \frac{k}{4}\int d^4x\, \epsilon^{\lambda\rho\mu\nu} (\partial_\lambda A_\rho)(\partial_\mu A_\nu)$

Right now the theory has a "topological" term, but it also has a metric dependent term. So at the moment the overall theory is not topological.

But... we can consider the relative importance of these terms in certain situations.

Big Big Step: In quantum mechanical theories, as we consider physics at different energy scales, we find that various "constants" e.g. masses, couplings, etc. actually flow, i.e. they are functions of the energy scale with which we probe the system (textbook values are at $E \sim 0$). There are technical ways to determine this flow (the entire process is called renormalization), but one rule of thumb that emerges is as follows.

First you evaluate the "length dimension" of a term in the action.
$\partial_\mu \sim \frac{1}{L}$, $A_\mu \sim \frac{1}{L}$ (since $D_\mu = \partial_\mu - A_\mu$)
then terms w/ higher length dimension dominate over terms of lower length dimension as we consider physics at low energies (or long length scales).

Comparing: $[S_{EM}] = \frac{1}{L^4}$, $[S_{top}] = \frac{1}{L^4}$ so both are always equally important.

However... if instead we consider $D = 2+1$, then the analogous story changes.

First of all $S_{EM} = -\frac{1}{4}\int d^3x \sqrt{-g}\, g^{\lambda\mu} g^{\rho\nu} F_{\lambda\rho} F_{\mu\nu}$ is unchanged (except $\mu,\nu,\lambda,\rho \in (0,1,2)$ and # of components change)

But now for S_{top} we need $\int A^{(3)}$ (instead of $A^{(4)}$). Clearly $F^{(2)} \wedge F^{(2)}$ will not do.
The simplest choice is:

$$S_{top} = \int d^3x\, \epsilon^{\lambda\mu\nu} A_\lambda F_{\mu\nu}$$

However now we find that $[S_{top}] = \frac{1}{L^3}$ while $[S_{EM}]$ is <u>still</u> $\frac{1}{L^4}$.

So in 2+1 we now find that this theory $S = S_{EM} + S_{top}$ at low energies (or long distance) is dominated by the S_{top} term... it has become purely topological!

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But what would this look like?

Take electrons, confine them to a 2D surface and then turn on a strong $\vec{B}$-field $\perp$ to the surface. The resulting theory is described by $S = S_{EM} + S_{top}$ w/ long distance physics governed by S_{top} (which is referred to as the Chern-Simons term, making this a Chern-Simons theory).

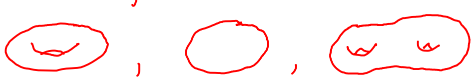
What would you calculate w/ it? Well first of all, since this theory does not depend on the metric $g_{\mu\nu}$, then $T^{\mu\nu} \propto \frac{\delta S}{\delta g_{\mu\nu}} = 0$, i.e. the energy-mom. vanishes!

But if $T^{\mu\nu} = 0$ then the Hamiltonian vanishes.

This is an awesome problem for introductory QM, solve $H\Psi = E\Psi$ when $H = 0$.

The answer of course is $E = 0$, but the nontrivial part of course is to find the wavefunctions Ψ, or at very least determine how many of them there are, i.e. the ground state degeneracy.

So what we find in this example is that the QM calculation of the g.s.d. in this system is effectively a calculation of the topology of the 2+1 spacetime (more interestingly the 2D space part since time is trivial). Of course this is interesting because (unlike in 3+1) we can actually create 2+1 spacetimes w/ interesting topologies, i.e.



etc.